Output controllability

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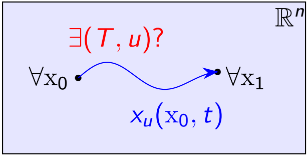
Consider for $t \geq 0$ the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t). \qquad (\star_S)$$

What is state controllability?



$\Longrightarrow$ Knowledge of the state at time T,

$$x_u(\mathrm{x}_0, T) = e^{TA}\mathrm{x}_0 + \int_0^T e^{(T-t)A} Bu(t)\mathrm{d}t.$$

How to characterize this notion?

Theorem (Kalman[1], Hautus[2])

The system $(\star_S)$ is state controllable if and only if :

1. $\exists\ T > 0$ such that $E^s_T(u) = \int_0^T e^{(T-\tau)A} Bu(\tau)\mathrm{d}\tau$, with $u \in L^\infty([0, T]; \mathbb{R}^m)$, is surjective.

[1] R. E. Kalman. "Contributions to the theory of optimal control". *Bol. Soc. Mat. Mexicana (2)* 5 (1960)

[2] M. L. J. Hautus. "Controllability and observability conditions of linear autonomous systems". *Nederl. Akad. Wetensch. Proc. Ser. A72* (1969)

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2. $\mathrm{rk}\left[B|AB|A^2B|\cdots|A^{n-1}B\right] = n$.
3. $\mathcal{G}_T^{\mathrm{s}} := \int_0^T e^{\tau A} B B^\top e^{\tau A^\top} \mathrm{d}\tau > 0$, for some time $T > 0$.

[1] R. E. Kalman. "Contributions to the theory of optimal control". *Bol. Soc. Mat. Mexicana (2)* 5 (1960)

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3. $\mathcal{G}^{\mathrm{s}}_T := \int_0^T e^{\tau A} BB^\top e^{\tau A^\top} \mathrm{d}\tau > 0$, for some time $T > 0$.
4. $\ker(B^\top) \cap \ker(A^\top - \lambda I_n) = \{0\}, \forall \lambda \in \mathbb{C}$.
5. $\operatorname{rk}(A - \lambda I_n | B) = n, \forall \lambda \in \mathbb{C}$.

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Theorem (Kalman[1])

If system $(\star_S)$ is state controllable (SC), then for every $(x_0, x_1, T) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ the control

$$u(t) = B^\top e^{(T-t)A^\top} \left(\mathcal{G}_T^{\mathrm{s}}\right)^{-1} \left(x_1 - e^{TA} x_0\right),$$

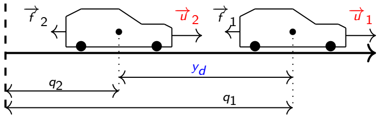
steers x_0 to x_1 in time $T > 0$. This control is the unique minimizer of

$$\min \quad \frac{1}{2} \int_0^T |u(t)|_m^2 \mathrm{d}t$$
$$\left| \begin{array}{l} u \in L^2([0,T]; \mathbb{R}^m), \\ x_1 = e^{TA} x_0 + \int_0^T e^{(T-t)A} B u(t) \mathrm{d}t. \end{array} \right.$$

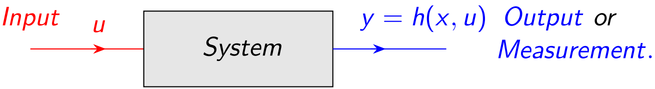
[1] R. E. Kalman. "Contributions to the theory of optimal control". *Bol. Soc. Mat. Mexicana (2)* 5 (1960)

Motivations

1. What if you don't want to control the hole state?



2. What if you have no information on the state of the system except probably its initial value?



Question:

Can we reach any benchmark output y_{ref} starting from any initial state data in finite time? If YES, what is the suitable control u to be used?

FRAMEWORK: Linear Time Invariant (LTI) systems

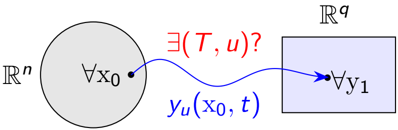
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \qquad (\star_O)$$

where, for $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$.

STRUCTURE

- Problem statement and state of art,
- Main results (contributions),
- Illustration of the results on an example,
- Conclusion.

Definition (State to output controllability)



u continuous
&
$y_u(\mathrm{x}_0, T) = \mathrm{y}_1$

Theorem (Kreindler & Sarachik[2])

The system $(\star_O)$ is state to output controllable if and only if:

(a) $\operatorname{rk}\left(CB|CAB|CA^2B|\cdots|CA^{n-1}B|D\right) = q.$

(b) $\mathcal{K}_T = \displaystyle\int_0^T Ce^{tA}BB^\top e^{tA^\top}C^\top \mathrm{d}t + DD^\top > 0$, *for some* $T > 0$.

[2] E. Kreindler and P. Sarachik. "On the concepts of controllability and observability of linear systems". *IEEE Transactions on Automatic Control* 9.2 (1964)

What do we want to do ?

We aim to

- extend the Hautus-Popov-Belevich criteria,
- establish a Gramian matrix condition that leads to a continuous control, when fulfilled.

Theorem (Danhane, Lohéac and Jungers[3])

The system $(\star_O)$ is state to output controllable if and only if:

1. There exists a time $T > 0$ such that the linear map E_T^o defined for every $u \in \mathcal{C}^0([0, T]; \mathbb{R}^m)$ by

$$E_T^o(u) = \int_0^T Ce^{(T-\tau)A}Bu(\tau)\mathrm{d}\tau + Du(T),$$

is surjective.

2. $\mathcal{G}_T^o := \int_0^T H_o(T,t)H_o(T,t)^\top \mathrm{d}t > 0$, for some $T > 0$, where

$$H_o(T,t) = \int_t^T Ce^{(T-\tau)A}B\mathrm{d}\tau + D.$$

[3] B. Danhane, J. Lohéac, and M. Jungers. "Characterizations of output controllability for LTI systems". *submitted, hal.03083128* (2020).

Theorem (Danhane, Lohéac and Jungers[3])

3. $\mathrm{rk}\,(C|D) = q$ and $\mathrm{Im}\begin{bmatrix} C^\top \\ D^\top \end{bmatrix} \cap \left(\bigoplus_{\lambda\in\sigma(A)} E_\lambda \times \{0\}\right) = \{0\}$, where $E_\lambda = \ker(A_\lambda^\top)^{n_\lambda} \cap \left(\bigcap_{k=0}^{n_\lambda-1} \ker B^\top (A_\lambda^\top)^k\right)$, $A_\lambda = A - \lambda I_n$, with n_λ the algebraic multiplicity of λ in the minimal polynomial of A.

4. $\mathrm{rk}\,(C|D) = q$ and
$$\mathrm{rk}\begin{bmatrix} K_{\lambda_1} & 0 & \cdots & 0 & (C|D)^\perp \\ 0 & K_{\lambda_2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & K_{\lambda_p} & (C|D)^\perp \end{bmatrix} = (n+m)p, \text{ where}$$
$p = \#\sigma(A)$, $\{\lambda_1, \lambda_2, \cdots, \lambda_p\} = \sigma(A)$, $K_\lambda = \begin{bmatrix} M_\lambda & 0 \\ 0 & I_m \end{bmatrix}$ and $M_\lambda = \left(A_\lambda^{n_\lambda} | A_\lambda^{n_\lambda-1} B | \cdots | A_\lambda B | B\right)$.

[3] B. Danhane, J. Lohéac, and M. Jungers. "Characterizations of output controllability for LTI systems". *submitted, hal.03083128* (2020).

Skech of the proof

- STEP 1: Reduction of the system.

Lemma (1)

Consider for $t \geq 0$, the system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ \dot{u}(t) &= v(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \quad \overset{\tilde{x}^\top = (x^\top, u^\top)^\top}{\Longleftrightarrow} \quad \begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}v(t) \\ y(t) &= \tilde{C}\tilde{x}(t), \end{aligned} \qquad (\tilde{\star}_O)$$

where $v(t) \in \mathbb{R}^m$ is the input, $\tilde{x}(t) \in \mathbb{R}^{n+m}$, $y(t) \in \mathbb{R}^q$, with $\tilde{A} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$ and $\tilde{C} = (C|D)$. System $(\star_O)$ is SOC if and only if system $(\tilde{\star}_O)$ is SOC.

Proof: It suffices to note that
$\mathcal{R}_o(\mathrm{x}_0, T) = \{Ce^{AT}\mathrm{x}_0\} + \mathrm{Im}\left(CB|CAB|CA^2B|\cdots|CA^{n-1}B|D\right)$ and
$\tilde{\mathcal{R}}_o(\tilde{\mathrm{x}}_0, T) = \{\tilde{C}e^{T\tilde{A}}\tilde{\mathrm{x}}_0\} + \mathrm{Im}\left(CB|CAB|CA^2B|\cdots|CA^{n-1}B|D\right)$.

Skech of the proof

- STEP 2: Prove the theorem with $D = 0$.

Lemma (2)

Assume that $D = 0_m^q$. The following conditions are equivalent:

1. *The system $(\star_O)$ is state to output controllable,*
2. $\operatorname{rk} C = q$ *and* $\operatorname{Im} C^\top \cap \bigoplus_{\lambda \in \sigma(A)} E_\lambda = \{0\}$,
3. $\operatorname{rk} C = q$ *and* $\operatorname{rk} \begin{bmatrix} M_{\lambda_1} & 0 & \cdots & 0 & C^\perp \\ 0 & M_{\lambda_2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & M_{\lambda_p} & C^\perp \end{bmatrix} = np.$

- STEP 3: Apply the result obtained in STEP 2: to system $(\tilde{\star}_O)$.

Proof of Lemma 2

$\star$**Hautus (1)**

$$\text{System } (\star_O) \text{ SOC} \quad \Leftrightarrow \quad \operatorname{rk}\left[CB|CAB|CA^2B|\cdots|CA^{n-1}B\right] = q,$$
$$\Leftrightarrow \quad \operatorname{rk} C = q \text{ and } \operatorname{Im} C^\top \cap \mathcal{N} = \{0\}, \text{ where}$$

$\mathcal{N} = \left\{\nu \in \mathbb{R}^n \mid {A^i}^\top \nu \in \ker B^\top,\ \forall i \in \mathbb{N}\right\}$. Finally, write $\mathcal{N} = \bigoplus_{\lambda\in\sigma(A)} \left(\mathcal{N} \cap \ker\left(A_\lambda^\top\right)^{n_\lambda}\right)$ [5] and show that $\mathcal{N} \cap \ker\left(A_\lambda^\top\right)^{n_\lambda} = E_\lambda$.

$\star$**Hautus (2)**

- $E_\lambda = \ker M_\lambda^\top$, where $M_\lambda = \left(A_\lambda^{n_\lambda}|A_\lambda^{n_\lambda-1}B|\cdots|A_\lambda B|B\right)$,
- Observe that $\operatorname{Im} C^\top = (\ker C)^\perp = \ker\left((C^\perp)^\top\right)$ for any full rank matrix $C^\perp$ satisfying $CC^\perp = 0$. $\square$

[5] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant subspaces of matrices with applications.* 2006

Theorem (Danhane, Lohéac and Jungers[3])

Let $(\mathrm{x}_0, \mathrm{y}_1) \in \mathbb{R}^n \times \mathbb{R}^q$*, and assume that system* $(\star_O)$ *is SOC. For every* $T > 0$ *and* $\mathrm{u}_0 \in \mathbb{R}^m$*, the control*

$$u(t) = \mathrm{u}_0 + \int_0^t H_o(T,\tau)^\top \mathrm{d}\tau (\mathcal{G}_T^o)^{-1} \left(\mathrm{y}_1 - y_{\mathrm{u}_0}(\mathrm{x}_0, T)\right),$$

steers x_0 *to* y_1 *in time* T*, where* $y_{\mathrm{u}_0}(\mathrm{x}_0, T) = Ce^{TA}\mathrm{x}_0 + H_o(T,0)\mathrm{u}_0$*.*

Furthermore, this control is the unique minimizer of

$$\min \quad \frac{1}{2}\int_0^T |\dot{u}(t)|_m^2 \mathrm{d}t \qquad \left| \begin{array}{l} u \in H^1([0,T];\mathbb{R}^m),\ u(0) = \mathrm{u}_0, \\ \mathrm{y}_1 = y_u(\mathrm{x}_0, T). \end{array} \right. \tag{Min}$$

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[3] B. Danhane, J. Lohéac, and M. Jungers. "Characterizations of output controllability for LTI systems". *submitted, hal.03083128* (2020)

Control with matrix $\mathcal{K}_T$

We also observe that form [4]

$$u(t) = \begin{cases} B^\top e^{(T-t)A^\top} C^\top (\mathcal{K}_T)^{-1} \delta_{\mathrm{y}} & \text{if } t \in [0, T), \\ D^\top (\mathcal{K}_T)^{-1} \delta_{\mathrm{y}} & \text{if } t = T, \end{cases}$$

with $\delta_{\mathrm{y}} = \mathrm{y}_1 - Ce^{AT}\mathrm{x}_0$ steers x_0 to y_1 in time T.
This control is the unique minimizer of

$$\min \quad \frac{1}{2}\int_0^T |u(\tau)|_m^2 \mathrm{d}\tau + \frac{1}{2}|z|_m^2$$

$$\left| \begin{array}{l} u \in L^2([0,T];\mathbb{R}^m), \; z \in \mathbb{R}^m, \\ \mathrm{y}_1 - Ce^{AT}\mathrm{x}_0 = CE_T^s(u) + Dz. \end{array} \right.$$

[4] E. Kreindler and P. Sarachik. "On the concepts of controllability and observability of linear systems". *IEEE Transactions on Automatic Control* 9.2 (1964)

Motion of two cars of masses m_1 and m_2

Take for instance

$$y(t) = (q_1(t) - q_2(t),\ v_1(t) - v_2(t),\ \dot{v}_1(t) - \dot{v}_2(t))^\top.$$

Setting $x = (q_1, v_1, q_2, v_2)^\top$, $u = (u_1, u_2)^\top$ and applying the FPD, the state variable x coupled with the output variable y yield $(\star_O)$ with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\alpha_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{\alpha_2}{m_2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -\frac{\alpha_1}{m_1} & 0 & \frac{\alpha_2}{m_2} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & -\frac{1}{m_2} \end{pmatrix}.$$

$m_1 = m_2 = 1$ and $\alpha_1 = \alpha_2 = 0.5$

- The SOC follows from the fact that the pair $(A,\, B)$ is state controllable and $\mathrm{rk}(C|D) = 3$.

- We will to steer $\mathrm{x}_0 = (1\ 0\ 1\ 0)^\top$ to $\mathrm{y}_1 = (2\ 0\ 0)^\top$ in time $T = 1$.

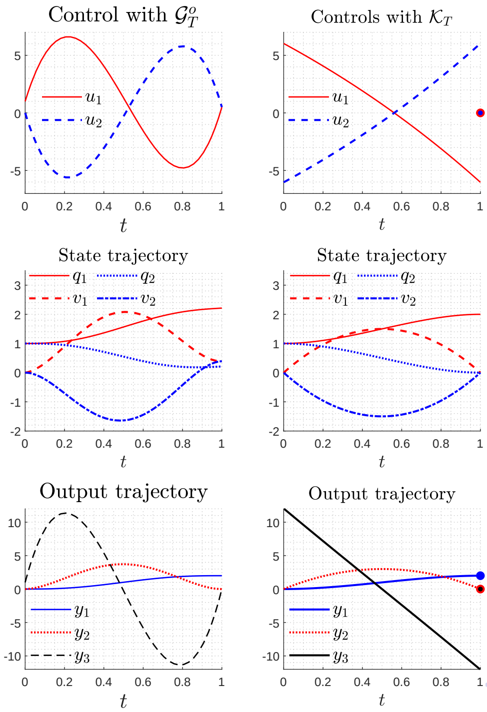
  $\star$ With $\mathrm{u}_0 = (1\ 0)^\top$ and $\mathcal{G}_1^o$ we get $u = (u_1,\, u_2)^\top$, with

$$\begin{aligned} u_1(t) &= 1 - at - bt^2 - c(e^{\frac{t}{2}} - 1),\ t \in [0,1] \\ u_2(t) &= 1 - u_1(t),\ t \in [0,1]. \end{aligned}$$

  $\star$ With $\mathcal{K}_1$ we have $u = (u_1,\, u_2)^\top$, with

$$u_1(t) = \begin{cases} g_1 e^{(t-1)/2} + g_2, & \forall t \in [0,1), \\ 0 & \text{if } t = 1, \end{cases}$$

$$u_2(t) = \begin{cases} -g_1 e^{(t-1)/2} - g_2, & \forall t \in [0,1), \\ 0 & \text{if } t = 1, \end{cases}$$


Control with $\mathcal{G}_T^o$
Controls with $\mathcal{K}_T$
u_1
u_2
State trajectory
State trajectory
q_1 q_2
v_1 v_2
Output trajectory
Output trajectory
y_1
y_2
y_3
t

Contributions

As contributions for LTI systems, we have

- extended the Hautus state controllability tests to the case of state to output controllability,
- given a Gramian criterion which, once fulfilled, leads to a computation of a continuous control and therefore a continuous output trajectory for any desire transfer,
- introduced two other notions of output controllability that can be found in:
  B. Danhane, J. Lohéac, and M. Jungers. "Characterizations of output controllability for LTI systems". *submitted, hal.03083128* (2020).