Quadratic behaviors of the 1D linear Schrödinger equation, with bilinear control

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Introduction to control theory.

$$\begin{cases} \frac{dx}{dt} = f(x, u), \quad t \in (0, T) \\ x(0) = x_0 \end{cases} \tag{1}$$

where, at time t,

- $x(t) \in \mathbb{R}^n$: **state** of this system,
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Definition

The system is said to be **controllable** at time T if for all initial conditions $x_0 \in \mathbb{R}^n$, for all targets $x_f \in \mathbb{R}^n$, there exists a control $u \in L^\infty(0, T)$ such that $x(T) = x_f$ where x is the solution of (2)

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Definition (E-STLC)

Let $(E_T, \|\cdot\|_{E_T})$ be a family of normed vector spaces of scalar functions defined on $[0, T]$ for $T > 0$. The system is said to be **E-controllable** at time T if for all $\varepsilon > 0$, for all initial conditions $x_0 \in \mathbb{R}^n$, for all targets $x_f \in \mathbb{R}^n$, there exists a control $u \in E_T$ with $\|u\|_{E_T} < \varepsilon$ such that $x(T) = x_f$ where x is the solution of (2)

When the linearized system is not controllable ?

Example 1:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^2. \end{cases} \qquad \begin{cases} \dot{y}_1 = u, \\ \dot{y}_2 = 0, \end{cases}$$

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- For the nonlinear system:

$$x_2(T) = \int_0^T u_1(t)^2 dt \geqslant 0$$

$\rightarrow$ no controllability for the nonlinear system.

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Example 2:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^3, \end{cases} \qquad \begin{cases} \dot{y}_1 = u, \\ \dot{y}_2 = 0, \end{cases}$$

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- For the nonlinear system: $(x_1(T), x_2(T)) = (a, b)^T$ iff

$$\begin{cases} u_1(T) = a, \\ \int_0^T u_1(t)^3 dt = b, \end{cases}$$

$\rightarrow$ controllability for the nonlinear system.

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Example 3:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3 + x_2^2. \end{cases} \qquad \begin{cases} x_1 = u_1, \\ x_2 = u_2, \\ x_3(T) = \int_0^T \left(u_1(t)^3 + u_2(t)^2\right) dt. \end{cases}$$

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First option. The quadratic term wins when $\|u'\|_{L^\infty(0,T)}$ **small**

$$\int_0^T u_1^3(t)dt = -\int_0^T u_2(t)2u(t)u_1(t)dt = \int_0^T u_2(t)^2 u'(t)dt$$

Then,

$$\begin{aligned} x_3(T) &\geqslant \left(1 - \|u'\|_{L^\infty(0,T)}\right) \int_0^T u_2(t)^2 dt \\ &> 0 \end{aligned}$$

$\rightarrow$ no controllability.

When the linearized system is not controllable ?

Second option. The **cubic** term **wins** for controls of the form:

$$u_\lambda(t) = \sqrt{\lambda}\phi''\left(\frac{t}{\lambda}\right).$$

Size of the control:

$$\|u_\lambda\|_{L^\infty(0,T)} \approx \sqrt{\lambda}, \quad \|u_\lambda'\|_{L^\infty(0,T)} \approx \frac{1}{\sqrt{\lambda}}.$$

Then,

$$\begin{aligned} x_3(T) &= \int_0^T u_1(t)^3 dt + \int_0^T u_2(t)^2 dt \\ &= \lambda^{\frac{11}{2}} \int_0^1 \phi'(\theta)^3 d\theta + \lambda^6 \int_0^1 \phi(\theta)^2 d\theta \\ &= a + o(a) \end{aligned}$$

with $\int_0^1 \phi'(\theta)^3 d\theta = sign(a)$ and $\lambda = |a|^{\frac{2}{11}}$.

$\rightarrow$ controllability

When the linearized system is not controllable ?

Second option. The **cubic** term **wins** for controls of the form:

$$u_\lambda(t) = \lambda^\alpha \phi''\left(\frac{t}{\lambda}\right).$$

Size of the control:

$$\|u_\lambda\|_{L^\infty(0,T)} \approx \lambda^\alpha, \quad \|u_\lambda'\|_{L^\infty(0,T)} \approx \lambda^{\alpha-1}.$$

Then,

$$\begin{aligned} x_3(T) &= \int_0^T u_1(t)^3 dt + \int_0^T u_2(t)^2 dt \\ &= \lambda^{3\alpha+4} \int_0^1 \phi'(\theta)^3 d\theta + \lambda^{2\alpha+5} \int_0^1 \phi(\theta)^2 d\theta \\ &= a + o(a) \end{aligned}$$

with $\alpha < 1$, $\int_0^1 \phi'(\theta)^3 d\theta = sign(a)$ and $\lambda = |a|^{1/3\alpha+4}$.

$\rightarrow$ controllability

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- Understand how the linearized test work
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  2. When controls are big in regular space, the **cubic** term allows us to gain back **controllability**.

Schrödinger equation.

Let $T > 0$.

$$\begin{cases} i\partial_t \psi(t,x) = -\partial_x^2 \psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, \quad t \in (0,T). \end{cases}$$

Bilinear control system

- the state: ψ, such that $\|\psi(t)\|_{L^2(0,1)} = 1$ for all time,
- $\mu : (0,1) \to \mathbb{R}$ dipolar moment of the quantum particle
- and $u : (0,T) \to \mathbb{R}$ denotes a scalar control.

Question.

$$\begin{cases} i\partial_t \psi = -\partial_x^2 \psi - u(t)\mu(x)\psi, & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T). \end{cases} \quad (3)$$

Definition (Small-time controllability around the ground state.)

*Let $(E_T, \|\cdot\|_{E_T})$ be a family of normed vector spaces of scalar functions defined on $[0,T]$, for $T > 0$. The system (3) is said to be **E-STLC around the ground state** if:*

$$\exists s \in \mathbb{N}, \quad \forall T > 0, \quad \forall \varepsilon > 0, \quad \exists \delta > 0,$$

$$\forall \psi_f \in \mathcal{S}, \|\psi_f - \psi_1(T)\|_{H^s_{(0)}(0,1)} < \delta, \quad \exists u \in L^2(0,T) \cap E_T,$$

$$\|u\|_{E_T} < \varepsilon, \quad \psi(0) = \varphi_1, \quad \psi(T) = \psi_f.$$

When the linear system is controllable.

$$i\partial_t \psi(t,x) = -\partial_x^2 \psi(t,x) - u(t)\mu(x)\psi(t,x), \quad \psi(t,0) = \psi(t,1) = 0.$$

Theorem (Beauchard and Laurent, 2010)

Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that

$$\exists c > 0 \text{ such that }, \forall j \in \mathbb{N}^*, |\langle \mu\varphi_1, \varphi_j \rangle| \geq \frac{c}{j^3}.$$

Then, the system is controllable in $\mathcal{S} \cap H^3_{(0)}$, locally around the ground state in arbitrary time $T > 0$ with controls in $L^2((0,T),\mathbb{R})$.

- Controllability of a linearized system : **moment problem**.
- Local controllability of the non-linear system : **inverse mapping** theorem to the end-point map.

Moment method.

First-order: $i\partial_t \Psi = -\partial_x^2 \Psi - u(t)\mu(x)\psi_1$
Explicit solution:

$$\Psi(T) = i \sum_{j=1}^{+\infty} \left(\langle \mu\varphi_1, \varphi_j \rangle \int_0^T u(t) e^{i(\lambda_j - \lambda_1)t} dt \right) \varphi_j e^{-i\lambda_j T}, \quad t \in (0, T).$$

- If $\langle \mu\varphi_1, \varphi_K \rangle = 0$, then

$$\langle \Psi(t), \varphi_K \rangle \equiv 0.$$

- If for all $j \in \mathbb{N}^*$, $\langle \mu\varphi_1, \varphi_j \rangle \neq 0$, the equality $\Psi(T) = \psi_f$ is equivalent to

$$\int_0^T u(t) e^{i(\lambda_j - \lambda_1)t} dt = -i \frac{\langle \psi_f, \varphi_j \rangle}{\langle \mu\varphi_1, \varphi_j \rangle} e^{i\lambda_j T}, \quad \forall j \in \mathbb{N}^*.$$

Theorem (The n-th quadratic obstruction.)

Let $n \in \mathbb{N}$ and $\mu \in H^{2n+1}((0,1),\mathbb{R})$ satisfying $\langle \mu\varphi_1, \varphi_K \rangle = 0$ for some $K \in \mathbb{N}^*$ + (Hyp)$_n$. The Schrödinger equation is not H^{2n-3}-STLC.

More precisely, there exists a time $T^* > 0$ such that for any final time $T \in (0, T)$, there exists a bound $\eta > 0$ such that for all controls $u \in H^{2n-3}(0,T)$ with $\|u\|_{H^{2n-3}(0,T)} \leq \eta$, if the solution ψ of Schrödinger with initial data φ_1 satisfied

$$\langle \psi(T), \varphi_{j_p} \rangle = 0 \quad \text{for } j_1, \dots, j_n \text{ s.t. } \langle \mu\varphi_1, \varphi_{j_p} \rangle \neq 0$$

then

$$\operatorname{Im}\left(\langle \psi(T), \varphi_K e^{-i\lambda_1 T} \rangle\right) \geq \alpha_K^n \int_0^T u_n(t)^2 dt \quad \text{with } \alpha_K^n > 0$$

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with $\alpha_K^n \approx \langle [\underbrace{[f_0,[f_0,\dots,[f_0,f_1]]}_{\text{length }(n,1)}, \underbrace{[f_0,[f_0,[f_0,\dots,[f_0,f_1]]]}_{\text{length }(n+1,1)}] (\varphi_1), \varphi_K\rangle$.

Idea of proof

Power series expansion: $\psi \approx \psi_1 + \Psi + \xi$.

- Linear term:

$$\langle \Psi(T), \varphi_K \rangle = 0.$$

- Quadratic term:

$$\mathrm{Im}\langle \xi(T), \varphi_K \rangle \geq 2\alpha_K^n \int_0^T u_n(t)^2 dt.$$

- Remainder:

$$\begin{aligned} \langle (\psi - \psi_1 - \Psi - \xi)(T), \varphi_K \rangle &= O\left(\|u_1\|^3_{L^2(0,T)} \right) \\ &= o\left(\int_0^T u_n(t)^2 dt \right) \end{aligned}$$

when $\|u\|_{H^{2n-3}(0,T)} \to 0$.

The next step

- The system is not H^{2n-3}-STLC.

- What happen for controls less regular ? Can the **cubic term prevails** on the quadratic term and thus allows to gain back **controllability** lost at the linear level, **despite a drift** ?

  Conjecture: the system is H^{2n-4}-STLC.

Toward a positive result

- On $\text{Span}(\varphi_K)$, **no controllability** at the linear level.
- On $H = \text{Span}(\varphi_K)^\perp$, **controllability** at the linear level and thus at the non linear level.

Step 1: On $[0, T_1]$, long φ_K,

$$\langle \psi(T_1), \varphi_K \rangle = \langle \psi_f, \varphi_K \rangle + o(\psi_f),$$

Step 2: On $[T_1, T_2]$, on H,

$$\mathbb{P}_H \psi(T_2) = \mathbb{P}_H \psi_f.$$

It remains to prove that

$$\langle \psi(T_2), \varphi_K \rangle - \langle \psi(T_1), \varphi_K \rangle = o(\psi_f).$$

$\rightarrow$ need simultaneous estimates on the control.