Discrete and potential mean field games

A Cournot mean field game example

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Contents

Framework and contributions

- Discrete time and discrete space mean field games [GMS10]
- Potential (or variational) structure [LL06]
- Interactions through congestion γ or price P mechanism allowing hard or soft constraints

	Soft	Hard
$\boldsymbol{F}$	$\gamma = \nabla \boldsymbol{F}$ [LL06]	$\gamma \in \partial \boldsymbol{F}$ [San12]
ϕ	$P = \nabla \phi$ [BHP21]	$P \in \partial \phi$ [GS20]

- Numerical methods : ADMM, ADMG, proximal primal-dual methods (see [AL20] for a survey)

Mean field game system

$$
\begin{cases}
\text{(i)} & \begin{cases} u(t,x) = & \inf_{\rho \in \Delta(S)} \sum_{y \in S} \rho(y) \Big(c_{\gamma,P}(t,x,y,\rho) + u(t+1,y) \Big), \\ u(T,x) = & \gamma(T,x), \end{cases} \\
\text{(ii)} & \pi(t,x,\cdot) \in \arg\min_{\rho \in \Delta(S)} \sum_{y \in S} \rho(y) \Big(c_{\gamma,P}(t,x,y,\rho) + u(t+1,y) \Big), \\
\text{(iii)} & \begin{cases} m(t+1,x) = & \sum_{y \in S} m(t,y)\pi(t,y,x), \\ m(0,x) = & m_0(x), \end{cases} \\
\text{(iv)} & \gamma(s,\cdot) \in \partial F(s, m(s,\cdot)), \\
\text{(v)} & P(t) \in \partial \phi\Big(t, \boldsymbol{Q}[m,\pi](t)\Big).
\end{cases}
\quad \text{(MFG)}
$$

Interpretation: individual player minimization problem

For any $(\gamma, P) \in \mathcal{U}$, we define the individual cost $c \colon \mathcal{T} \times S \times S \times \Delta(S) \to \mathbb{R}$,

$$c_{\gamma,P}(t,x,y,\rho) = \ell(t,x,\rho) + \gamma(t,x) + \alpha(t,x,y)P(t).$$

Given $(m,\pi) \in \mathcal{R}$, we denote the aggregated demand

$$\boldsymbol{Q}[m,\pi](t) = \sum_{(x,y)\in S^2} m(t,x)\pi(t,x,y)\alpha(t,x,y).$$

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The dynamical system of each agent is a Markov chain $(X_s^\pi)_{s\in\bar{\mathcal{T}}}$ controlled by $\pi \in \Delta$, with initial distribution m_0: for any $(t,x,y) \in \mathcal{T} \times S^2$,

$$\mathbb{P}\left(X_{t+1}^\pi = y | X_t^\pi = x\right) = \pi(t,x,y), \quad \mathbb{P}(X_0^\pi = x) = m_0(x).$$

Given the coupling terms $(\gamma, P) \in \mathcal{U}$, the individual control problem is

$$\inf_{\pi\in\Delta} J_{\gamma,P}(\pi) := \mathbb{E}\left(\sum_{t\in\mathcal{T}} c_{\gamma,P}(t, X_t^\pi, X_{t+1}^\pi, \pi(t,X_t^\pi)) + \gamma(T, X_T^\pi)\right).$$

The mean field game problem is given by:

$$\pi \in \arg\min_{\rho\in\Delta} J_{\gamma,P}(\rho), \quad \gamma \in \partial \boldsymbol{F}[m^\pi], \quad P \in \partial\phi[\boldsymbol{Q}[m^\pi,\pi]].$$

Contents

Primal and dual potential problems

- Primal problem :

$$\inf_{(m,w)\in\mathcal{R}} \tilde{\mathcal{J}}(m,w) := \sum_{(t,x)\in\mathcal{T}\times S} \tilde{\ell}[m,w](t,x) + \sum_{t\in\mathcal{T}} \phi[\boldsymbol{A}w](t) + \sum_{s\in\bar{\mathcal{T}}} \boldsymbol{F}[m](s), \quad (\tilde{P})$$

subject to: $\boldsymbol{S}w - m + \bar{m}_0 = 0.$

- Dual problem : We define the mapping $\boldsymbol{U}\colon \mathcal{U} \to \mathbb{R}(\bar{\mathcal{T}} \times S)$ associates with $(\gamma, P) \in \mathcal{U}$ the solution $u \in \mathbb{R}(\bar{\mathcal{T}} \times S)$ to the dynamic programming equation

$$\begin{cases} u(t,x) + \ell^\star[-\boldsymbol{A}^\star P - \boldsymbol{S}^\star u](t,x) = \gamma(t,x) & (t,x) \in \mathcal{T} \times S, \\ u(T,x) = \gamma(T,x), & x \in S. \end{cases}$$

$$\max_{(\gamma,P)\in\mathcal{U}} \tilde{\mathcal{D}}(\gamma,P) := \langle \bar{m}_0, \boldsymbol{U}[\gamma,P]\rangle - \sum_{t\in\mathcal{T}} \phi^\star[P](t) - \sum_{s\in\bar{\mathcal{T}}} \boldsymbol{F}^\star[\gamma](s). \qquad (\tilde{D})$$

Main results

Assumptions

- Maps $\ell(t,x,\cdot)$, $F(s,\cdot)$, and $\phi(t,\cdot)$ are proper, convex and lower semi-continuous. In addition $\text{dom}(\ell(t,x,\cdot)) \subseteq \Delta(S)$ is convex.
- Qualification Assumption.

Results

- Duality result : $\min\ (\tilde{P}) = -\max\ (\tilde{D})$.
- Let (m, π, u, γ, P) be a solution to (MFG). Then $(m, w := m\pi)$ is solution to $(\tilde{P})$ and (γ, P) is solution to $(\tilde{D})$.
- Let (m, w) be solution to $(\tilde{P})$ and (γ, P) be solution to $(\tilde{D})$. Let $\hat{u} = \boldsymbol{U}[\gamma, P]$ and let $\pi \in \boldsymbol{\pi}[m, w, \hat{u}, \gamma, P]$. Then $(m, \hat{\pi}, \hat{u}, \gamma, P)$ is a solution to (MFG).
- There exists a solution (m, π, u, γ, P) to (MFG). If $F(s,\cdot)$ and $\phi(t,\cdot)$ are differentiable then (u, γ, P) is uniquely defined. If $\ell(t,x,\cdot)$, $F(s,\cdot)$, and $\phi(t,\cdot)$ are strictly convex, (MFG) has a unique solution.

Contents

Situation (with and without hard constraint)

- The state $x \in S$ of an individual agent represents a level of stock
- $\alpha(t, x, y) = y - x$ represents the quantity bought in order to "move" from a level of stock x to y.
- The individual cost $\ell(t, x, \rho) = \langle \rho(t, x), \beta(t, x) \rangle$ for some matrix β, is linear.
- The potential $\phi[D] = \phi_1[D] + \phi_2[D]$, where

$$\phi_1[D] = \frac{1}{4}(D + \bar{D})^2, \quad \phi_2[D] = \chi_{(-\infty, D_{\max}]}(D).$$

  The potential ϕ is the sum of a convex and differential term ϕ_1 with full domain and a convex non-differentiable term ϕ_2.
- The level of demand

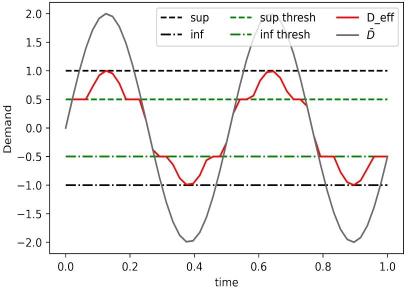
$$D(t) = \boldsymbol{Q}[m, \pi](t) = \sum_{(x,y) \in S^2} m(t, x)\pi(t, x, y)\alpha(t, x, y).$$

  The quantity $\bar{D}$ is a given exogenous quantity which represent a net demand (positive or negative) to be satisfied by the agents. In this example $\bar{D}(t) = 2\sin(4\pi t/(T - 1))$ for any $t \in \mathcal{T}$ and $D_{\max} = 0$.

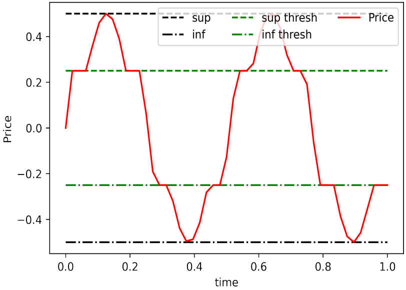
Cournot mean field game system

$$
\begin{cases}
\text{(i)} & \begin{cases} u(t,x) = & \inf_{\rho \in \Delta(S)} \ell(t,x,\rho) + \sum_{y \in S} \rho(y)(\alpha(t,x,y)P(t) + u(t+1,y)), \\ u(T,x) = & 0, \end{cases} \\
\text{(ii)} & \quad \pi(t,x,\cdot) \in \underset{\rho \in \Delta(S)}{\arg\min} \ \ell(t,x,\rho) + \sum_{y \in S} \rho(y)(P(t)\alpha(t,x,y) + u(t+1,y)), \\
\text{(iii)} & \begin{cases} m(t+1,x) = & \sum_{y \in S} m(t,y)\pi(t,y,x), \\ m(0,x) = & m_0(x), \end{cases} \\
\text{(iv)} & \quad P(t) \in \partial\phi\,(t, D(t))\,.
\end{cases}
$$

Solution of the game without hard constraints: $\phi_2 = 0$

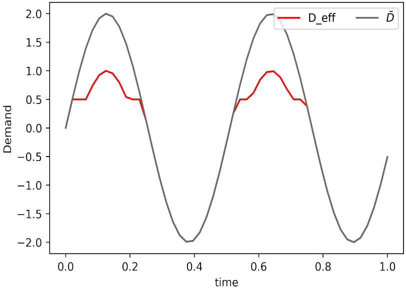


(a) $\bar{D}$ and effective demand D_{eff}

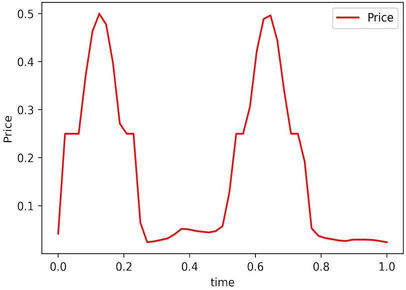


(b) Price P

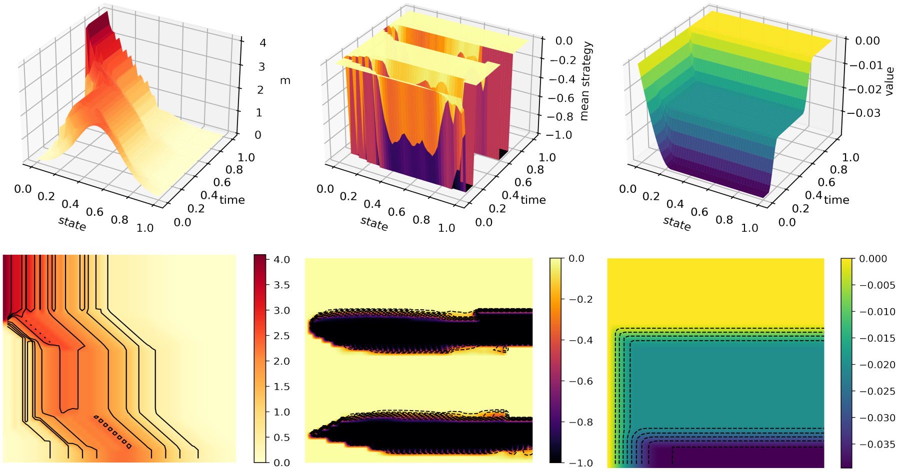
Solution of the game with hard constraints: $\phi_2[D] = \chi_{(-\infty,0]}(D)$



(c) $\bar{D}$ and effective demand D_{eff}



(d) Price P



(h) Measure m

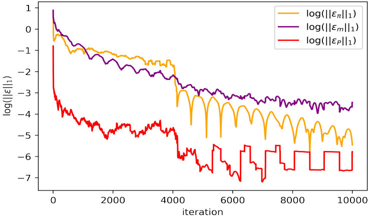
(i) Mean displacement

(j) Value u

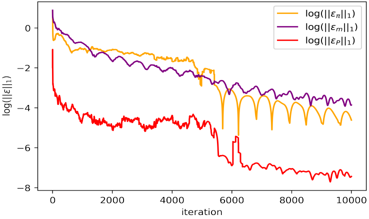
$$v(t,x) := \sum_{y \in S} \pi(t,x,y)(y-x)$$

Contents

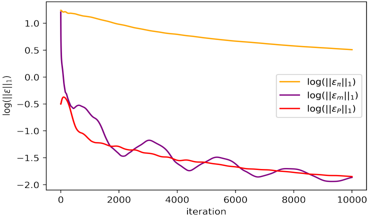
Summary about the numerical methods



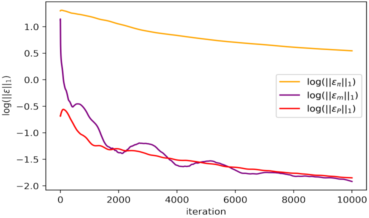
(k) ADMM



(l) ADMG



(m) Chambolle-Pock



(n) Chambolle-Pock-Bregman

Method	Convergence guarantee	Execution time (s)
ADMM	No	2000
ADMG	Yes	2000
Chambolle-Pock	$\mathcal{O}(1/k)$	1600
Chambolle-Pock-Bregman	$\mathcal{O}(1/k)$	1300

Figure: Convergence guarantee and execution time, $N = 10000, |\mathcal{T}| = |S| = 50$

Contents

Conclusion

- General framework for discrete and potential mean field games
- Existence and uniqueness results
- Numerical experiments with 4 methods
- Code available at lavignepierre.github.io
- Paper [BLP21] available on ArXiv

Thank you for your attention

References I

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