Single-pass computation of first travel time for seismic waves in 3D TTI media

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Outline

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First arrival time

- The **eikonal equation** characterizes the first arrival time of a wave front propagating inside a domain at a speed determined by a given metric.
- In geophysics, the eikonal equation can be obtained as the high-frequency approximation of the **elastic wave equation**, with the metric defined by the elastic properties of the geological medium.

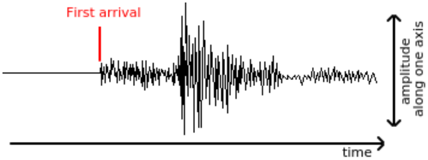

Figure 1: Seismogram and first arrival time (of the P wave)

- We face specific challenges when solving the eikonal equation, with difficulties arising in anisotropic 3D media (Le Bouteiller et al., 2019; Desquilbet et al., 2020).

- A typical model for the elastic properties of a geophysical medium is the TTI (tilted transverse isotropic) model: we suppose that the elastic properties of the medium are the same in any direction perpendicular to a symmetry axis.
- A TTI medium can naturally occur from isotropic sedimentary layers which can be treated as anisotropic for seismic waves propagating at wavelengths much larger than the layer thicknesses.

Figure 2: Sedimentary layers

Outline

The first arrival time u is a viscosity solution to the TTI equation, which comes from a high-frequency approximation of the elastic wave equation, of the form:

$$\boxed{ap_r^4 + bp_z^4 + cp_r^2 p_z^2 + dp_r^2 + ep_z^2 = 1}$$

where p is the slowness vector: $(p_x, p_y, p_z) = R\nabla u$, with R the rotation of the TTI medium, and $p_r^2 = p_x^2 + p_y^2$.

The parameters (a, b, c, d, e) are determined from the Thomsen parameters $(V_p, V_s, \epsilon, \delta)$ as:

$$\begin{cases} a &= -(1+2\varepsilon)V_p^2 V_s^2 \\ b &= -V_p^2 V_s^2 \\ c &= -(1+2\varepsilon)V_p^4 - V_s^4 + (V_p^2 - V_s^2)(V_p^2(1+2\delta) - V_s^2) \\ d &= V_s^2 + (1+2\varepsilon)V_p^2 \\ e &= V_p^2 + V_s^2 \end{cases}$$

Slowness surfaces

TTI equation: $\boxed{ap_r^4 + bp_z^4 + cp_r^2 p_z^2 + dp_r^2 + ep_z^2 = 1}$

- The TTI equation actually has several solutions, corresponding to the propagation speeds of the P-wave (the fastest) and of the S-wave.
- We call **slowness surfaces** the solutions to the TTI equation with parameter p: the interior surface corresponds to the P-wave, and the exterior surface corresponds to the S-wave.

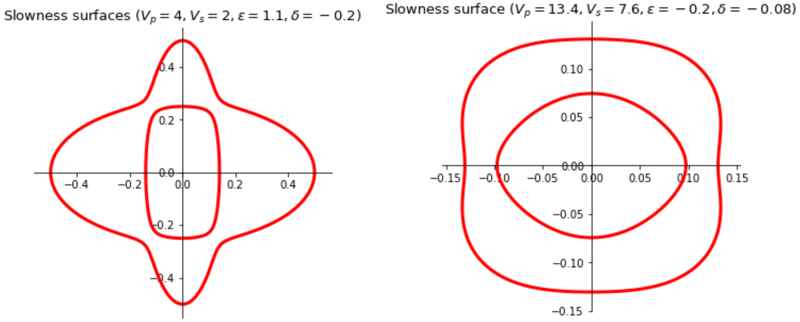

Figure 3: Slowness surfaces for two TTI media in the (p_r, p_z) domain

- A **Riemannian metric** is a metric for which the slowness surface is an ellipse, and eikonal equations with Riemannian metrics can be solved efficiently with tools from (Mirebeau, 2014).
- For TTI media, the idea is to approximate the P-slowness surfaces by ellipses either by the outside or by the inside, and locally consider the TTI metric as an optimization problem over Riemannian metrics.

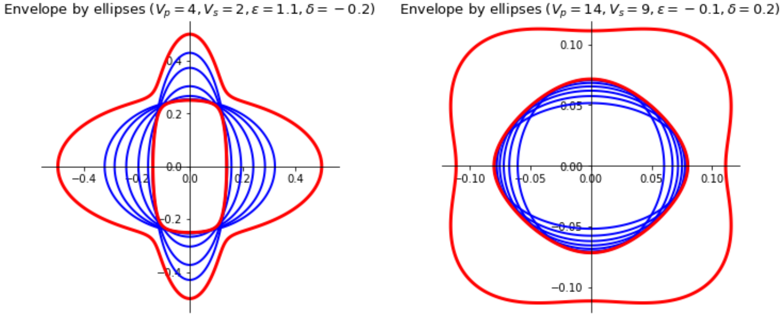

Figure 4: Envelope by ellipses of the P-slowness surfaces

Obtaining the envelope by ellipses

The TTI equation can be considered in the “root domain”:

$$\boxed{ap_r^4 + bp_z^4 + cp_r^2 p_z^2 + dp_r^2 + ep_z^2 = 1} \text{ becomes } \boxed{aq_r^2 + bq_z^2 + cq_r q_z + dq_r + eq_z = 1}$$

with $q_r = p_r^2$, $q_z = p_z^2$.

- The second equation is the equation of a conic, for which we can calculate tangential straight lines.
- In the (p_r, p_z) domain, the straight lines become ellipses, which correspond to Riemannian metrics.

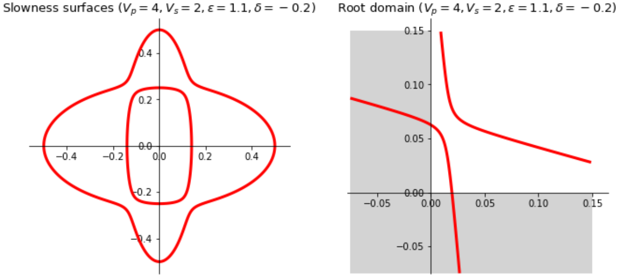

Figure 5: Example of tangential ellipse and representation in the root domain

Obtaining the envelope by ellipses

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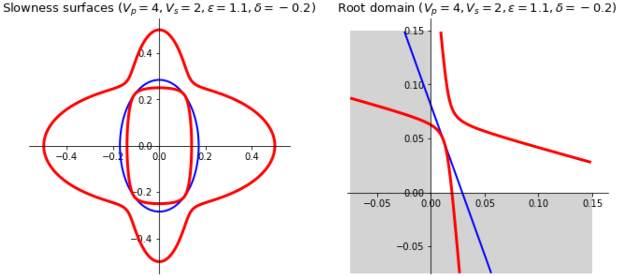

Figure 6: Example of tangential ellipse and representation in the root domain

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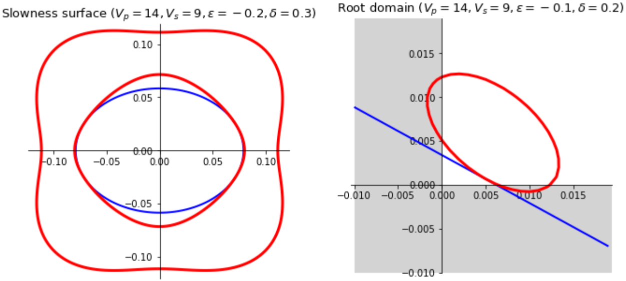


Figure 7: Example of tangential ellipse and representation in the root domain

Remarks on the envelope

- For a slowness surface, the conic in the root domain is either a hyperbola or an ellipse, which leads to an envelope by ellipses of the P-surface either by the outside or by the inside.
- Therefore, we can consider the TTI metric as either a minimum or a maximum over Riemannian metrics.

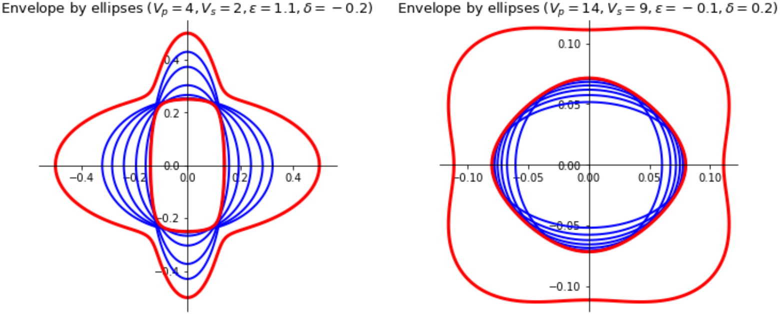
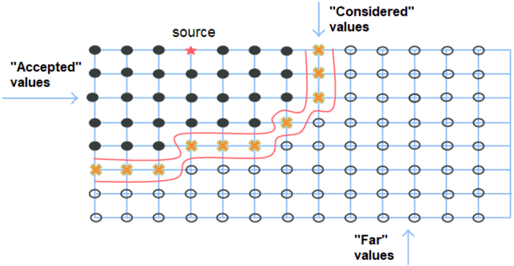


Figure 8: Envelope by ellipses

Outline

- The Fast Marching method is a **single-pass method**, of complexity $n\log(n)$, similar to the Dijkstra algorithm, in which we follow the propagation of a wavefront inside the discretized domain (Sethian, 1996; Tsitsiklis, 1995).
- The arrival time at each point is computed by a **local update operator** Λ, which estimates the arrival time at a position x from the arrival times at its neighbours y:

$$\boxed{u(x) = \Lambda[u(y), y \text{ neighbours of } x]}$$



- A Riemannian metric in dimension n is locally characterized by a symmetric positive definite matrix $D \in S_n^{++}$, with the corresponding eikonal equation: $||\nabla u(x)||_{D(x)} = 1$.
- Assume that we have the decomposition: $D = \sum_{i=1}^{d} \rho_i e_i e_i^T$, with $\rho_i > 0$ and $e_i \in \mathbb{Z}^n$ (such a decomposition can be obtained in dimension 3 with the Selling decomposition).
- Then we can consider the numerical scheme on the Cartesian grid with grid size h:

$$||\nabla u(x)||^2 = \sum_{i=1}^{d} \rho_i \max\{0, \frac{u(x) - u(x \pm he_i)}{h}\}^2 + \mathcal{O}(h)$$

- From this, we can define the local update operator Λ by:

$$\Lambda u(x) = \lambda \Leftrightarrow \sum_{i=1}^{d} \rho_i \max\{0, \frac{\lambda - u(x \pm he_i)}{h}\}^2 = 1$$

1D optimization problem over Riemannian metrics

In the case of a maximum (resp. minimum) over a set of Riemannian metrics indexed by $k \in \mathcal{K}$, we modify the local update operator as:

$$\boxed{u(x) = \max_{k \in \mathcal{K}} \Lambda_k[u(y), y \text{ neighbours of } x]}$$

$$(\text{resp. } u(x) = \min_{k \in \mathcal{K}} \Lambda_k[u(y), y \text{ neighbours of } x])$$

We considered two methods to solve the optimization problem:

- Exhaustive grid-search over ellipses: not very precise, but can be done very efficiently with GPU acceleration
- Newton-like algorithm: the optimization problem is not globally convex (resp. concave), but it can be divided into a finite number of sections, and a search algorithm with an exponential convergence rate is possible in each section.

- The optimization problem over Riemannian metrics corresponds to an optimization over a segment with the mapping presented in Figure 9.

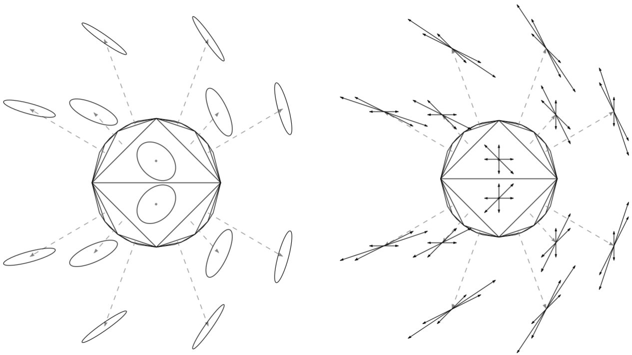


Figure 9: Left: Slowness surfaces of Riemannian metrics, with mapping: $\begin{pmatrix} 1+a & b \\ b & 1-a \end{pmatrix}$, $a^2 + b^2 < 1$
Right: Stencils used in the corresponding numerical scheme

- For a same set of stencils used in the numerical scheme, we showed that the optimization problem has at most one local extremum.

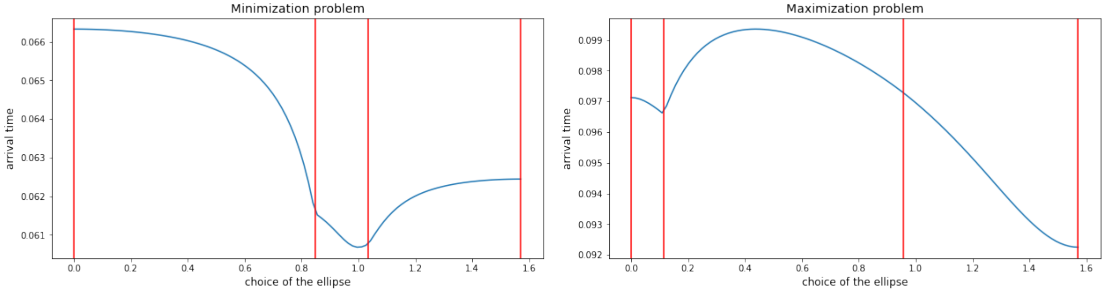


Figure 10: Examples of 1D optimization problems. The vertical red lines corresponds to the modifications of the stencils needed for the Riemannian scheme.

Outline

We want to study the performances of our numerical solver on a heterogeneous 3D TTI metric for which we know the exact solution.

- We know how to obtain the exact solution for any homogeneous metric.
- We know how conformal transformations modify both the metric and the space.

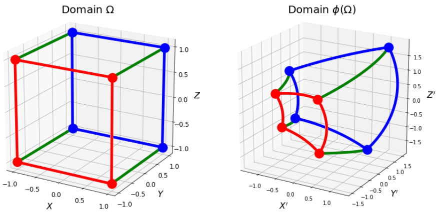


Figure 11: Image of a cube by a "special conformal transformation"

Heterogenous 3D metric with known exact solution

- From the conformal transformation on a homogeneous metric, we obtain a fully anisotropic heterogeneous medium.
- We can solve the corresponding eikonal equation with our numerical solver. (Figure, left)
- Besides, it also corresponds to a homogeneous metric when it is mapped by the conformal transformation, and we can compute the exact solution in this case. (Figure, right)

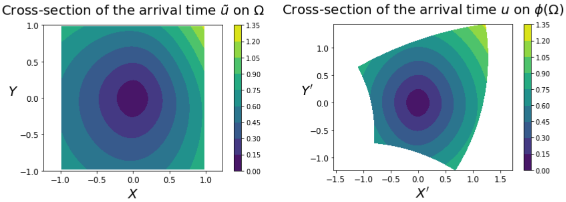


Figure 12: Cross-sections of the arrival time before and after mapping by the transformal conformation

- The L^2-error is of order 2 relatively to the grid step size, and computation time is quasi-linear relatively to the number of points (optimization method used: Newton-like algorithm).
- Proper attention to source factorization is needed to achieve order 2: we implemented additive source factorization and a multiscale strategy around the source point.
- Examples computed on a laptop with 32G RAM, Processor Intel Core i7-8665U CPU @ 1.90GHz x 8 (computation time for a 209 $\times$ 209 $\times$ 209 model: 4min 02s).

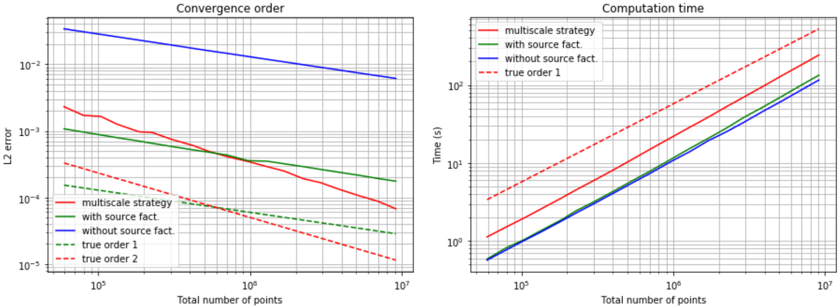


Figure 13: Error convergence and computation time

Outline

Conclusions and perspectives

Conclusion:

- I presented an algorithm to compute the first arrival time of seismic waves in 3D TTI media, with a single-pass approach using the Fast Marching method.
- We obtained second-order precision scheme and quasi-linear in computation time on smooth test-cases.

Perspectives:

- Comparison between the two methods for the 1D optimization problem.
- Comparisons with other numerical schemes for the TTI equation.
- Numerical applications on realistic test cases.
- Integration of the method in a scheme for tomography in seismic imaging.

References

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Thank you for your attention!